Integer solutions to the anomaly equations for a class of chiral gauge theories

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Based on:
AP and Filippo Revello — 2205.03428

The axion solution to the strong CP problem

- Experimental fact: absence of CP violation in the QCD sector
- Theoretical puzzle: QCD theta angle must be extremely small: $\theta < 10^{-10}$

nEDM - PRL 124 (2020) 8, 081803

- Axion solution:

Peccei and Quinn, Wilczek, Weinberg (1977) + […]

  - there is a spontaneously broken anomalous $U(1)_{PQ}$ symmetry
  - the axion is the Nambu-Goldstone boson associated to this symmetry
  - the axion potential generated by QCD effects relaxes θ to 0

$$V(a) = -m_\pi^2 f_\pi^2 \sqrt{1 - \frac{4m_u m_d}{(m_u + m_d)^2} \sin^2\left(\frac{a}{2f_a} - \frac{\theta}{2}\right)}$$

  - θ is dynamically relaxed to exactly 0

Vafa, Witten (1984)

The axion quality problem

- The $U(1)_{PQ}$ symmetry is a global symmetry
- But global symmetries are explicitly broken in quantum gravity
- PQ violating effects generated at the Planck scale can spoil the axion solution

UV effects : local PQ violating operator with dimension $\Delta_{\not{PQ}}$

↓

IR effect

$$\Delta\theta \approx |c|\, \phi_{CP} \left(\frac{M_{\rm Pl}}{\Lambda_{\rm QCD}}\right)^4 \left(\frac{f_a}{M_{\rm Pl}}\right)^{\Delta_{\not{PQ}}}$$

- neutron EDM experiment: $\Delta\theta \lesssim 10^{-10}$

high quality for $f_a \lesssim 10^{12}\,\mathrm{GeV}$ → $\Delta_{\not{PQ}} \gtrsim 12$

Kamionkowski and March-Russell, Barr and Seckel (1992) + [...]

Some proposed solutions

- Axions in string theory with exponentially small corrections

  - axions and axion-like particles arise naturally in string theory — Witten (1984)
  - $\delta V(a) \sim \Lambda_{UV}^4 \, e^{-S_{UV}} \, \cos\left(a/f_a + \delta\right)$
  - It is still unclear what are the necessary conditions to ensure small UV effects

Kallosh et al. (1995), Svrcek and Witten, Conlon (2006) — Demirtas et al. (2021), Heidenreich et al. (2021)

- Quantum field theories with fundamental scalar fields, extra-dimensions, etc.

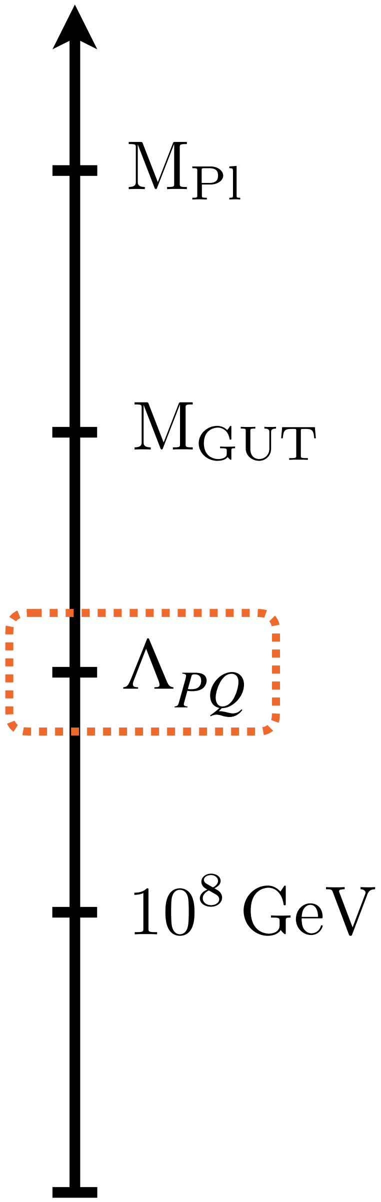
  - discrete gauge symmetries, extra-dimensions, new gauge interactions, …

    Barr and Seckel, Chun and Lukas (1992) + […]

- Quantum field theories with fundamental fermions and gauge dynamics in 4D

we focus on this class of models

Models of composite axions

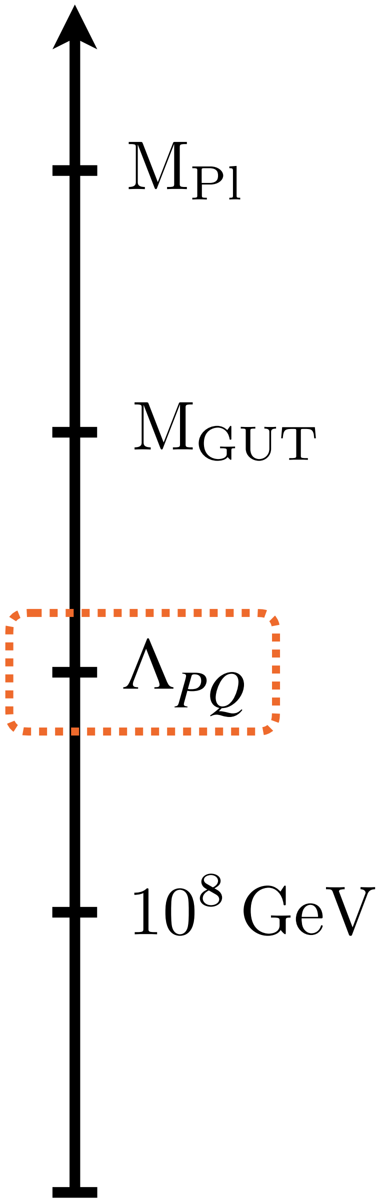


- $SU(N) \times G_W \times G_{SM}$

$$\psi_i \sim (\Box, p_i, r_i), \qquad \chi_i \sim (\bar{\Box}, q_i, \bar{r}_i), \qquad i = 1, \ldots, n_f$$

- r_i is a (possibly reducible) rep of G_{SM} or its GUT extension
- G_W is a generic compact Lie group assumed to be weak at Λ_{PQ}
- ψ_i , χ_i are left-handed Weyl fermions

Models of composite axions



- $SU(N) \times G_W \times G_{SM}$

$$\psi_i \sim (\square, p_i, r_i), \qquad \chi_i \sim (\bar{\square}, q_i, \bar{r}_i), \qquad i = 1, \ldots, n_f$$

- r_i is a (possibly reducible) rep of G_{SM} or its GUT extension
- G_W is a generic compact Lie group assumed to be weak at Λ_{PQ}
- ψ_i , χ_i are left-handed Weyl fermions
- simplest example:

	$SU(N)$	$SU(3)_c$	$U(1)_{PQ}$
ψ_1	$\square$	$\square$	$+1$
ψ_2	$\square$	$\mathbf{1}$	-3
χ_1	$\bar{\square}$	$\bar{\square}$	$+1$
χ_2	$\bar{\square}$	$\mathbf{1}$	-3

In this example:
fermionic mass terms
set to zero by hand

Kim - PRD 31 (1985) 1733

Selection rules on PQ violating operators

- Not every PQ violating operator is dangerous !
- A generic PQ violating operator generates a potential only if it has non vanishing matrix element with a state containing axions:

$$\langle \psi_a | \mathcal{O}_{\not{PQ}} | 0 \rangle \neq 0$$

- The operator must be an interpolating operator for the axion, with vanishing vectorial charges.

→ $\mathcal{O}_{\not{PQ}}$ polynomial in $(\psi_r \chi_{\bar{r}}),\ (\psi_r \chi_{\bar{r}})^*, (\psi_r^\dagger \psi_r)$ and $(\chi_{\bar{r}}^\dagger \chi_{\bar{r}})$

important!

- It can be a composite operator built from the insertion of N local operators

$$d_{\text{eff}} = \sum_{i=1}^{N} d_i - 4(N-1)$$

Models with an abelian factor

$$SU(N) \times U(1) \times G_{\rm SM}$$

$$\psi_i \sim (\Box, p_i, r_i), \qquad \chi_i \sim (\bar{\Box}, q_i, \bar{r}_i), \qquad i = 1, \ldots, n_f$$

- $SU(N) \times U(1) \times G_{\rm SM}$
- r_i is a (possibly reducible) rep of $G_{\rm SM}$ or its GUT extension
- p_i and q_i are integer charges for $U(1)_D$ with:

  $p_i \neq -q_i \qquad (i \neq j)$ → chiral charge assignment

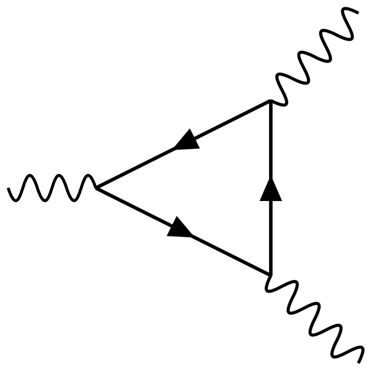
- The classification of PQ violating operators requires a knowledge of the U(1) charges

The anomaly cancellation equations

- Local gauge anomalies for $U(1)$ impose non-trivial constraints

$$\begin{cases} \sum_{i=1}^{n_f} (p_i + q_i) T_i = 0, & U(1) \times [G_{\rm SM}]^2 \\ \sum_{i=1}^{n_f} (p_i + q_i) d_i = 0, & U(1) \times [SU(N)]^2 \\ \sum_{i=1}^{n_f} (p_i^3 + q_i^3) d_i = 0, & [U(1)]^3 \end{cases}$$

(zero hypercharge for simplicity)



Dimension of SM rep:

$$d_i = \dim(r_i) = \sum_\alpha \dim(r_i^{(\alpha)}),$$

Dynkin index of SM rep:

$$T_i = \sum_\alpha T(r_i^{(\alpha)}),$$

The anomaly cancellation equations

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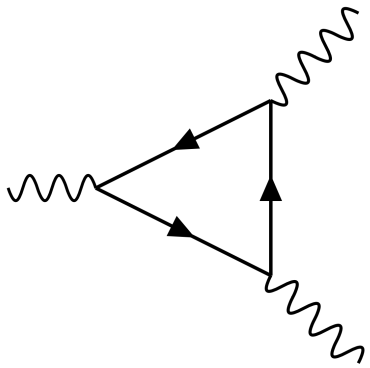


- System of polynomial equations over the integers
  - cubic equations: in general it is very hard to find all the integer solutions
  - The case of a pure U(1) gauge theory has been solved recently

Costa, Dobrescu, Fox '19, Allanach, Gripaios, Tooby-Smith '19

General solution for $n_f = 2$

- For $n_f = 2$ and chiral assignments $p_i \neq -q_i$

  - the two linear equations have to be equivalent (so $T_1 d_2 = T_2 d_1$).

$$\begin{cases} (p_1 + q_1)d_1 + (p_2 + q_2)d_2 = 0 \\ (p_1^3 + q_1^3)d_1 + (p_2^3 + q_2^3)d_2 = 0 \end{cases}$$

  - combining the two equations we obtain an homogeneous quadric

$$Q(X, Y, Z) \equiv (d_2^2 - d_1^2)X^2 + 3d_2^2 Y^2 - 3d_2^2 Z^2 = 0$$

Conic in projective space!

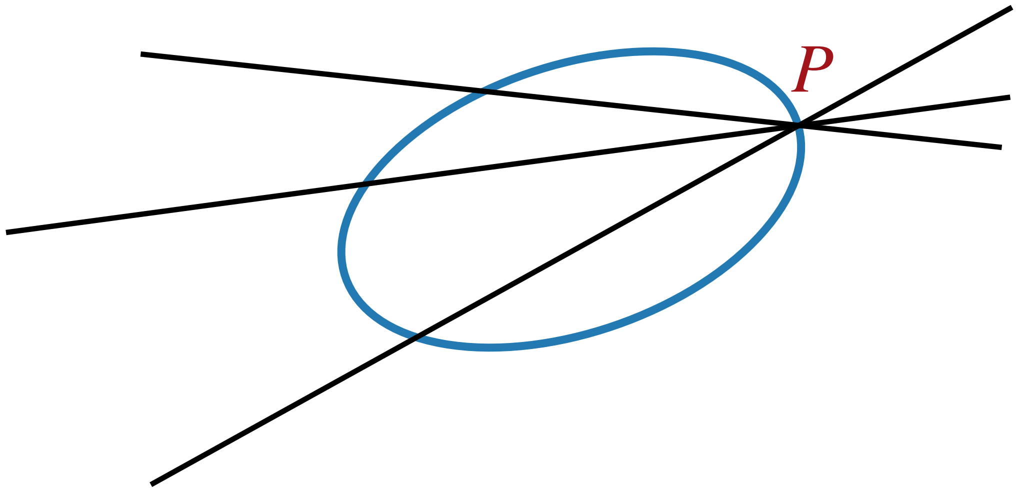
where $X = (p_1 + q_1), \quad Y = (p_1 - q_1), \quad Z = (p_2 + q_2)$

  - Integers charges correspond to the zero locus of $Q(X, Y, Z)$ in $\mathbb{PQ}^2$

General solution for $n_f = 2$

- Theorem 1 (see e.g. Mordell):

Given a rational point P on a conic, there are in fact infinitely many rational points and they can all be found as the intersection of a rational line through P and the conic itself.



- In our case $P = [0:1:1]$, and we find:

$$p_1 = \frac{n}{\mu_2}\tilde{p}_1 = \frac{n}{\mu_2}\Big[d_1^2\ell^2 + d_2^2(3k^2 + 6k\ell - \ell^2)\Big],$$

$$q_1 = \frac{n}{\mu_2}\tilde{q}_1 = \frac{n}{\mu_2}\Big[-d_1^2\ell^2 + d_2^2(\ell^2 + 6k\ell - 3k^2)\Big],$$

$$p_2 = \frac{n}{\mu_2}\tilde{p}_2 = \frac{n}{\mu_2}\Big[d_1^2\ell^2 - 6d_1d_2k\ell - d_2^2(3k^2 + \ell^2)\Big],$$

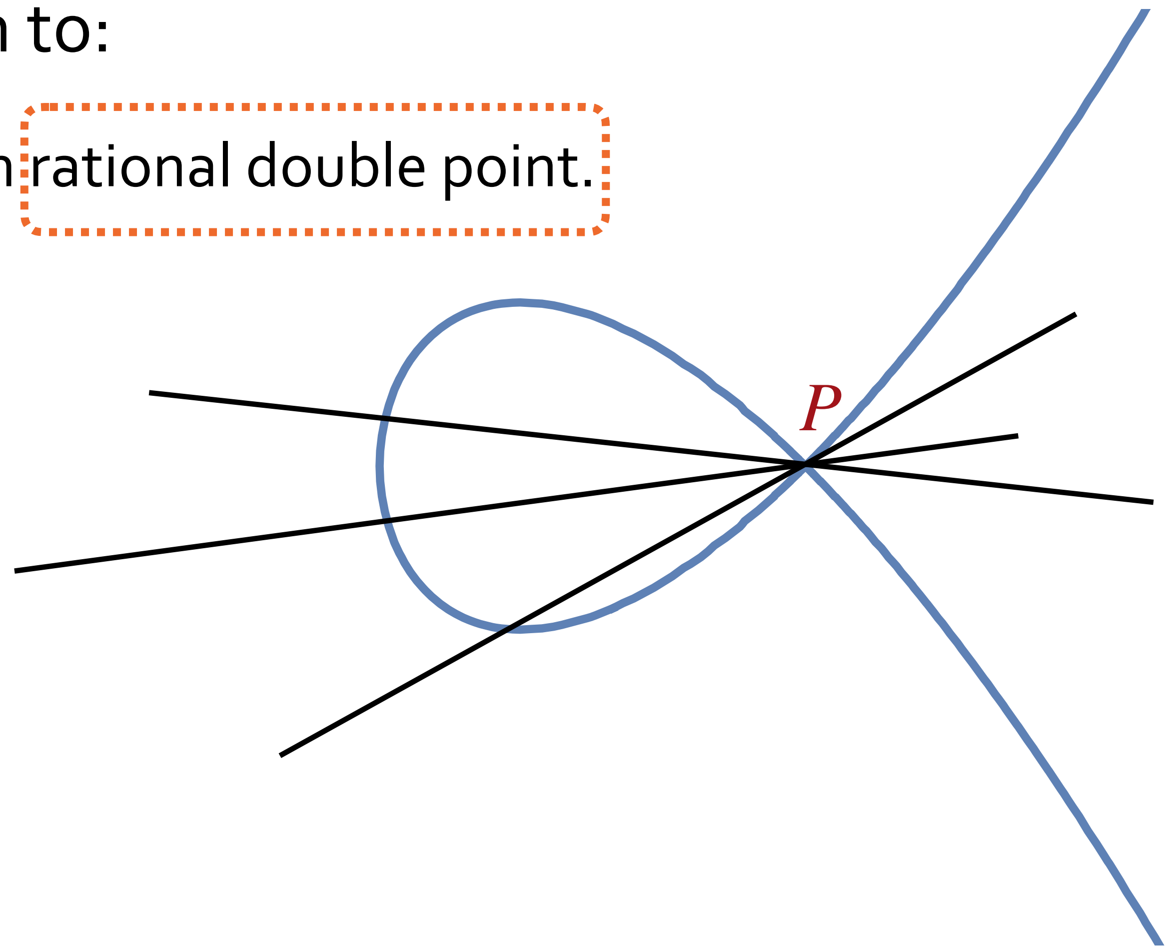
$$q_2 = \frac{n}{\mu_2}\tilde{q}_2 = \frac{n}{\mu_2}\Big[-d_1^2\ell^2 - 6d_1d_2k\ell + d_2^2(3k^2 + \ell^2)\Big],$$

$$n, k, \ell \in \mathbb{Z} \setminus \{0\}$$

$$\mu_2 = \gcd\Big(\tilde{p}_1, \tilde{p}_2, \tilde{q}_1, \tilde{q}_2\Big)$$

General solution for arbitrary n_f

- In the general case we cannot reduce the system to a quadric.
- However, for our specific system of equations, by using one of the linear equations, we can reduce the system to:
  - a singular cubic hypersurface, with a known rational double point.
  - plus an additional linear equation.



- Theorem 2 (see e.g. Mordell):

Given a rational double point P on a cubic hypersurface, there are infinitely many rational points and they can all be found as the intersection of a rational line through P and the cubic itself.

General solution for arbitrary n_f

- Setting $\nu = n_f$ for notational convenience, we find

$$p_i = \frac{n}{\mu_\nu}\tilde{p}_1 = \frac{n}{\mu_\nu}\Big[A_2 - A_1\,\ell_i\Big], \qquad i = 1,\ldots,\nu{-}1,$$
$$q_i = \frac{n}{\mu_\nu}\tilde{q}_1 = \frac{n}{\mu_\nu}\Big[-A_2 - A_1\,m_i\Big],$$
$$p_\nu = \frac{n}{\mu_\nu}\tilde{p}_\nu = \frac{n}{\mu_\nu}\Big[A_2 - A_1\,\ell_\nu\Big],$$
$$q_\nu = \frac{n}{\mu_\nu}\tilde{q}_\nu = \frac{n}{\mu_\nu}\Big[-A_2 + \frac{A_1}{d_\nu}\Big(\sum_{i=1}^{\nu-1} d_i(\ell_i + m_i) + d_\nu\,\ell_\nu\Big)\Big],$$

  - charges parametrized by a set of arbitrary integers subject to a linear constraint:

$$\sum_{i=1}^{\nu-1} D_{i\nu}(\ell_i + m_i) = 0. \qquad \ell_i, m_i, n \in \mathbb{Z}$$

where $D_{ij} = d_i T_j - d_j T_i$, $\mu_\nu = \gcd(\tilde{p}_1,\ldots,\tilde{q}_\nu)$, A_I are polynomials in (ℓ_i, m_i)

General solution for arbitrary n_f

$$A_1 = 3d_\nu^2 \sum_{i=1}^{\nu-1} d_i(\ell_i^2 - m_i^2) + 3d_\nu^3\,\ell_\nu^2 - 3d_\nu \left(\sum_{i=1}^{\nu-1} d_i(\ell_i + m_i) + d_\nu\,\ell_\nu\right)^2 ,$$

$$A_2 = d_\nu^2 \sum_{i=1}^{\nu-1} d_i(\ell_i^3 + m_i^3) + d_\nu^3\,\ell_3^3 - \left(\sum_{i=1}^{\nu-1} d_i(\ell_i + m_i) + d_\nu\,\ell_\nu\right)^3 .$$

- the knowledge of the $U(1)$ charges allows us to identify the form of gauge invariant PQ violating local operators.

- For $\nu = 2$ and $\nu = 3$ the gauge invariance of PQ violating operators of the form:

$$\mathcal{O}_{\not{PQ}} = (\psi_1\chi_1)^{\kappa_1}\ldots(\psi_\nu\chi_\nu)^{\kappa_\nu}$$

is independent from the charge assignments — depends only on SM representations

Single insertions can be easily characterised → useful for phenomenology

Some open questions

- We found all the charge assignments such that local gauge anomalies cancel
  - what about global (non-perturbative) anomalies?
- Are there more efficient ways to classify charge assignments such that operators with a given property (e.g. PQ violating operators) are forbidden up to a given dimension?
- Do all the consistent charge assignments admit an embedding in string theory?